An Efficient High-Order Mixed-Edge Rectangular-Element Method for Lossy Anisotropic Dielectric Waveguides

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***Abstract*—A new efficient high-order mixed-edge rectangular-element method is proposed for the analysis of lossy anisotropic dielectric waveguides. The space construction of the high-order mixed-edge rectangular element is investigated and the explicit form of the shape function is given. The high-order mixed-edge element yields higher accuracy and faster convergence than the lowest order mixed-edge rectangular elements without spurious solutions, and is more efficient compared to the high-order covariant projection element. The computations of the propagation constants in the rectangular waveguide and the slab loaded waveguide show that the accuracy of this high-order mixed-edge element is about one order higher than that of the lowest order one, and the nodes used in the calculation are only two-thirds as many as those used in the high-order covariant projection element having the same accuracy. The calculations of the dispersion curves for the dominant mode in the waveguide loaded with the lossy anisotropic dielectric block verify the accuracy and efficiency of the present method.**

***Index Terms*— Anisotropic, dielectric waveguide, dispersion, mixed-edge rectangular-element method.**

I. Introduction

THE functional formulation with full vector $\mathbf{H}$ is widely used to rigorously evaluate propagation problems of different complicated guiding structures filled with isotropic or anisotropic dielectrics. The most serious problem associated with this approach is the appearance of the spurious solutions. It is recognized that the following two aspects may produce spurious solutions. First, conventional finite elements are nonconforming in the curl space. Nedelec has proved that the continuity of the tangential components between two adjacent elements is sufficient and a necessary condition of the conforming element in the curl space [10]. The additional forced continuity of the normal component in the conventional finite elements breaks the conforming in the curl space [14]. Secondly, conventional finite elements cannot correctly model the null space of the curl operator. The full vector $\mathbf{H}$ is composed of three components, and the polynomial orders of interpolated functions for each component in the functional must be matched in order to correctly model the null space of the curl operator.

The penalty-function method [2]–[4] has been used to cure this spurious modes problem, but in this technique an arbitrary positive constant, called the penalty coefficient, is involved and the accuracy of solutions depends on its magnitude. In recent years, the lowest order [12] and the high-order mixed-edge triangular element [13]–[16] have been developed to eliminate the spurious solutions successfully. In some cases, rectangular elements are suitable for modeling electromagnetic fields in block-shaped guiding structures, which are widely used in microwave circuit. In 1984, Hano proposed the lowest order mixed-edge rectangular in [17], however, the accuracy of the mixed-edge element is required to be improved for better applications. To this end, the covariant projection elements for vector finite-element problems have been developed in [18]. The high-order covariant projection element yields higher accuracy and faster convergence, but the number of nodes increases dramatically with division supplementing because many inner nodes exist in the element. This decreases the efficiency of this kind of edge-element.

In this paper, a new efficient high-order mixed-edge rectangular-element method is proposed. The space construction of the high-order mixed-edge rectangular element is investigated and the explicit form of the shape function is given. This element yields higher accuracy and faster convergence than the lowest order mixed-edge rectangular elements, and is more efficient compared to the high-order covariant projection element. The computations of the propagation constants in the rectangular waveguides and the slab-loaded waveguides show that the accuracy of the high-order mixed-edge element approach is about one order higher than that of the lowest order approach, and the nodes used in the analysis are only two-thirds as many as those used in the high-order covariant projection element with the same accuracy. The calculations of the dispersion curves for the dominant mode in the waveguide filled with lossy anisotropic dielectric segment verify the accuracy and efficiency of the present method.

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II. Analysis of High-Order Mixed-Edge Element

It is well known that the eigenvalue problem of guided-wave structures is equivalent to the variational problem of various

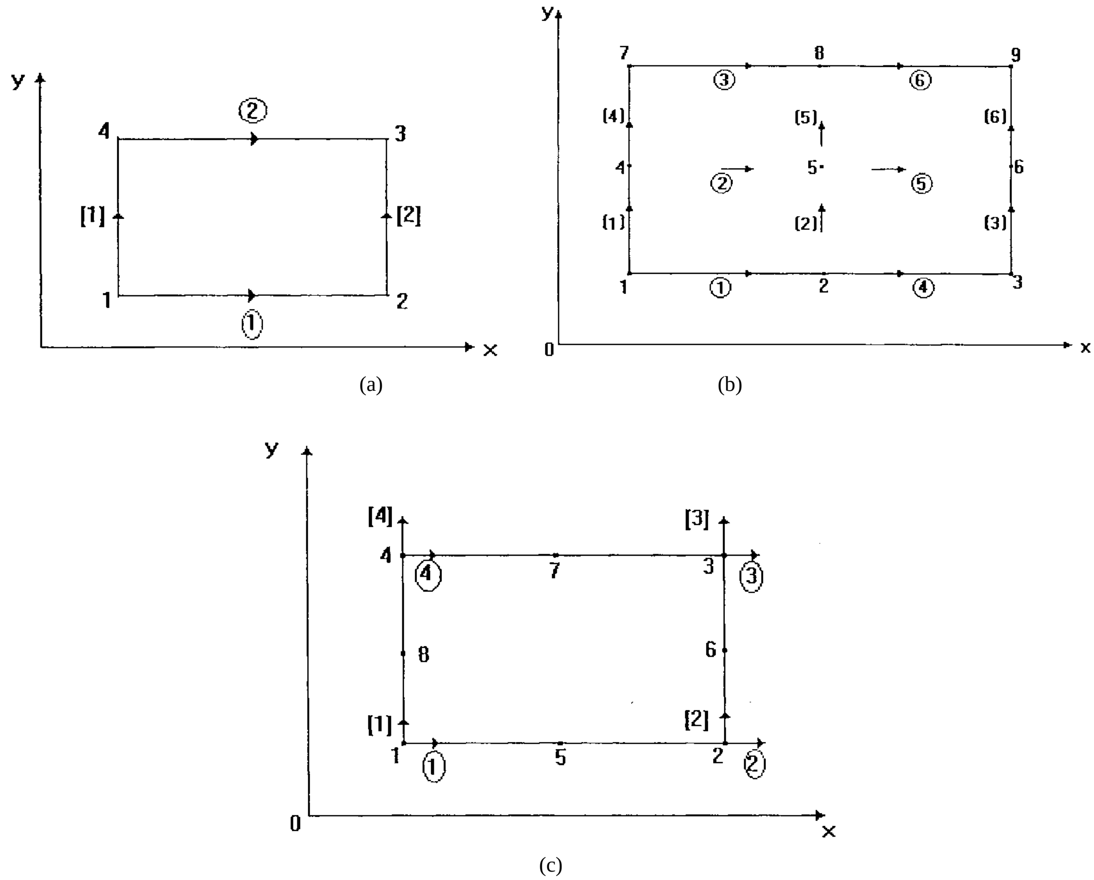


Fig. 1. (a) Lowest order mixed-edge rectangular element. (b) High-order covariant projection element. (c) High-order mixed-edge rectangular element.

functional formulations. It is justified by many of the practical experiences of the various formulations, with the following full-vector functional shown to be quite suitable for a lot of practical and complicated problems

$$F(\mathbf{H}) = \int\int_{\Omega} [(\nabla\times\mathbf{H})^{*}\cdot([p]\nabla\times\mathbf{H}) - k_0^2[q]\mathbf{H}^{*}\cdot\mathbf{H}]\,dxdy \quad (1)$$

where $[p]$ and $[q]$ are, respectively, the permittivity and permeability tensors, which are defined as

$$[p] = \begin{bmatrix} p_x & 0 & 0 \\ 0 & p_y & 0 \\ 0 & 0 & p_z \end{bmatrix}, \qquad p_x = \frac{1}{\epsilon_{rx}}, p_y = \frac{1}{\epsilon_{ry}}, p_z = \frac{1}{\epsilon_{rz}} \quad (2)$$

$$[q] = \begin{bmatrix} q_x & 0 & 0 \\ 0 & q_y & 0 \\ 0 & 0 & q_z \end{bmatrix}, \qquad q_x = \mu_{rx}, q_y = \mu_{ry}, q_z = \mu_{rz}. \quad (3)$$

When the finite-element method (FEM) is used to solve the above variational problem, one has to first of all, accurately and reasonably construct the finite-element space. To do this, one should keep two criterions in mind. One is that finite-element space must be conforming in the curl space, which can be satisfied only if tangential components are used as interpolated parameters. The other is that finite-element space should correctly model the null space of the curl operator. For analyzing propagation problems in the guiding structure, the curl operator can be described as

$$\nabla\times \mathrm{H} = \left(\frac{\partial H_z}{\partial y} + j\beta H_y\right)\hat{e}_x + \left(-j\beta H_x - \frac{\partial H_z}{\partial x}\right)\hat{e}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{e}_z. \quad (4)$$

Hence, the null space of the curl operator is determined by the following equations:

$$\frac{\partial H_z}{\partial y} + j\beta H_y = 0 \quad (5)$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = 0 \quad (6)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0. \quad (7)$$

From (5) to (7), one sees that in order to keep the same polynomial order for each term in the equations, which is required for correctly modeling the null space of the curl operator, the shape function of H_z should have one-order higher than that of H_x in x, one-order higher than that of H_y in y.

Fig. 1(a) shows the lowest order mixed-edge rectangular element proposed in [17], which is composed of two tangential unknowns (1–2) parallel to x-direction for interpolating the component H_x, two tangential unknowns 1–2 parallel to y-direction for interpolating the component H_y, and four

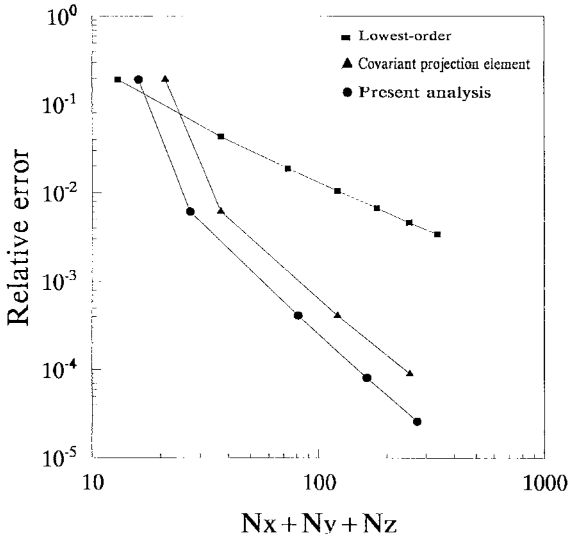


Fig. 2. Rate of convergence.

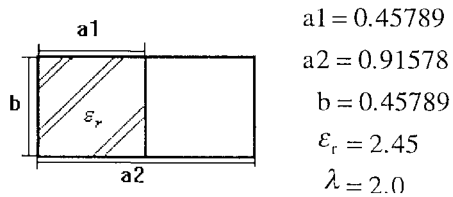


Fig. 3. The slab-loaded waveguide.

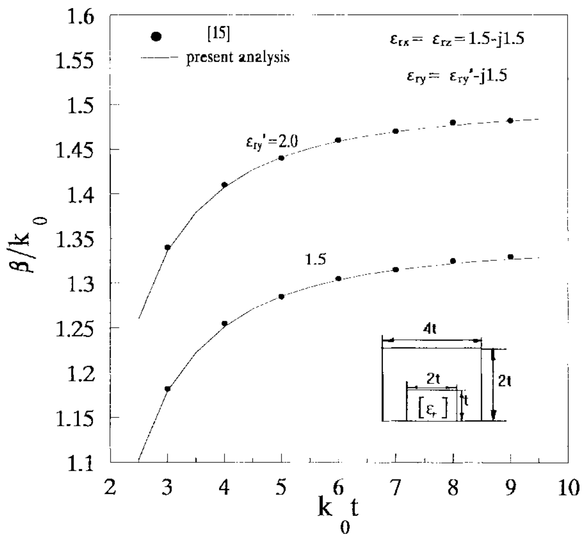
unknowns (1–4) parallel to z-direction for interpolating the axial component H_z. Obviously, this finite-element space satisfies the above two criterions.

Fig. 1(b) shows a kind of high-order mixed covariant projection element developed according to [18], which contains some inner nodes and is composed of six unknowns (1–6) parallel to x-direction for interpolating the component H_x, six unknowns (1–6) parallel to y-direction for interpolating the component H_y, and nine unknowns (1–9) parallel to z for interpolating the axial component H_z. This finite-element space also satisfies the above two criterions.

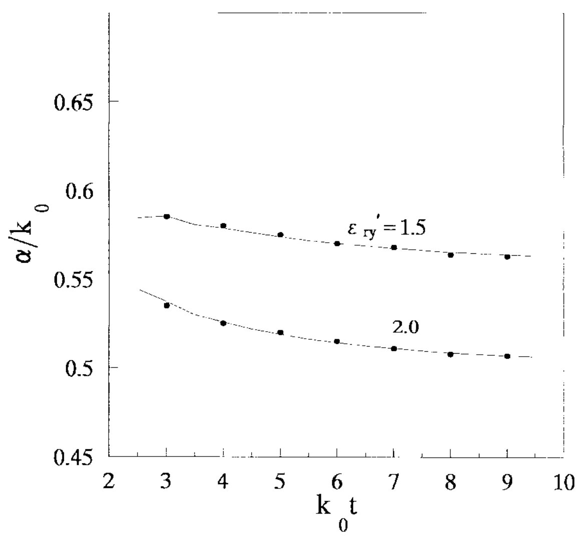
Here, a new high-order mixed edge rectangular element is proposed as shown in Fig. 1(c), which is composed of four tangential unknowns (1–4) parallel to x-direction for interpolating the component H_x, four tangential unknowns (1–4) parallel to y-direction for interpolating the component H_y, and eight unknowns (1–8) parallel to z-direction for interpolating the axial component H_z. It is easy to prove this new finite-element space satisfies the above two criterions. Since there are no inner nodes in the element, the total number of nodes is reduced and the computing efficiency of the method is improved. Some numerical results given in the next section confirm this conclusion.

In order to conveniently obtain interpolating function for $\mathbf{H}$, the x–y coordinate is transformed to ξ–η coordinate according to the following transformation formulas:

$$\xi = \frac{x - (x_2 + x_1)/2}{l_x}, \qquad l_x = \frac{x_2 - x_1}{2} \tag{8}$$



(a)



(b)

Fig. 4. Dispersion characteristics of the E_{11}^{y} mode in the lossy dielectric block-loaded waveguide. (a) Normalized phase constant. (b) Normalized attenuation constant.

$$\eta = \frac{y - (y_4 + y_1)/2}{l_y}, \qquad l_y = \frac{y_4 - y_1}{2}. \tag{9}$$

Hence, the following expressions for the electromagnetic field $\mathbf{H}$ in each element have been derived:

$$H_x^e = \sum_{i=1}^{4} U_i H_{xi} \tag{10}$$

$$H_y^e = \sum_{i=1}^{4} V_i H_{yi} \tag{11}$$

$$H_z^e = \sum_{i=1}^{8} j N_i H_i \tag{12}$$

with

$$U_1 = (1-\xi)(1-\eta)/4, \qquad V_1 = (1-\xi)(1-\eta)/4$$
$$U_2 = (1+\xi)(1-\eta)/4, \qquad V_2 = (1+\xi)(1-\eta)/4$$

TABLE I
A COMPARISON OF THE PROPAGATION CONSTANT OF THE SLAB LOADED WAVEGUIDES BETWEEN THE THEORETICAL AND CALCULATED RESULTS

Eigen modes	Theoretical value	present analysis (Nx+Ny+Nz=163)		high-order covariant projection element (Nx+Ny+Nz=253)		lowest-order (Nx+Ny+Nz=253)	
		calculated	error(%)	calculated	error(%)	calculated	error(%)
first	2.51331	2.51275	0.02	2.51227	0.04	2.49769	0.62
second	-5.41743	-5.43196	0.27	-5.42430	0.12	-5.51866	1.7
third	-5.55285	-5.55962	0.12	-5.55960	0.12	-5.65088	1.8

$$
\begin{aligned}
U_3 &= (1+\xi)(1+\eta)/4, \qquad V_3 = (1+\xi)(1+\eta)/4\\
U_4 &= (1-\xi)(1+\eta)/4, \qquad V_4 = (1-\xi)(1+\eta)/4\\
N_1 &= -(1-\xi)(1-\eta)(1+\xi+\eta)/4\\
N_2 &= -(1+\xi)(1-\eta)(1-\xi+\eta)/4\\
N_3 &= -(1+\xi)(1+\eta)(1-\xi-\eta)/4\\
N_4 &= -(1-\xi)(1+\eta)(1+\xi-\eta)/4\\
N_5 &= -(1-\xi^2)(1-\eta)/2\\
N_6 &= -(1+\xi)(1-\eta^2)/2\\
N_7 &= -(1-\xi^2)(1+\eta)/2\\
N_8 &= -(1-\xi)(1-\eta^2)/2.
\end{aligned}
$$

Substituting (10)–(12) into (1), one obtains the following final eigenvalue equations, which gives a solution directly for the propagation constant β:

$$
[A_{tt}]\begin{Bmatrix} H_x \\ H_y \end{Bmatrix} = \beta^2 [B_{tt}] \begin{Bmatrix} H_x \\ H_y \end{Bmatrix} \tag{13}
$$

where $[A_{tt}]$, $[B_{tt}]$, and $[A_{xx}]$ are shown at the bottom of the page, with

$$
\begin{aligned}
[A_{xy}] &= \sum_e \int\int_e [p_z\{U_y\}\{V_x\}^T]\,dxdy\\
[A_{xz}] &= \sum_e \int\int_e [p_y\{U\}\{N_x\}^T]\,dxdy\\
[A_{yy}] &= \sum_e \int\int_e [p_z\{V_x\}\{V_x\}^T - q_y k_0^2\{V\}\{V\}^T]\,dxdy\\
[A_{yz}] &= \sum_e \int\int_e [p_x\{V\}\{N_y\}^T]\,dxdy\\
[A_{zz}] &= \sum_e \int\int_e [p_y\{N_x\}\{N_x\}^T + p_x\{N_y\}\{N_y\}^T\\
&\qquad - q_z k_0^2\{N\}\{N\}^T]\,dxdy\\
[B_{xx}] &= \sum_e \int\int_e [p_y\{U\}\{U\}^T]\,dxdy\\
[B_{yy}] &= \sum_e \int\int_e [p_x\{V\}\{V\}^T]\,dxdy.
\end{aligned}
$$

III. NUMERICAL RESULTS

In order to check the accuracy of the new high-order mixed-edge rectangular elements, some numerical results are given here. Fig. 2 shows the relative error of the computed β for the dominant mode in the rectangular waveguide and the convergence property of numerical calculations, where N_x, N_y, and N_z are the numbers of the nodes for H_x, H_y, and H_z components, respectively, and $N_x+N_y+N_z$ corresponds to the total number of the degrees of freedom. The relative error is defined as

$$
\text{Relative error} = (\beta_{\text{exact}} - \beta_{\text{FEM}})/\beta_{\text{exact}}.
$$

It is confirmed from Fig. 2 that the high-order mixed-edge rectangular elements can give accurate results with faster convergence than the lowest mixed-edge rectangular elements and the high-order covariant projection element [18]. Table I

$$
[A_{tt}] = \begin{bmatrix} [A_{xx}] & [A_{xy}] \\ [A_{yx}] & [A_{yy}] \end{bmatrix}
$$

$$
[B_{tt}] = \begin{bmatrix} [B_{xx}] - [A_{xz}][A_{zz}]^{-1}[A_{zx}] & -[A_{xz}][A_{zz}]^{-1}[A_{zy}] \\ -[A_{yz}][A_{zz}]^{-1}[A_{zx}] & [B_{yy}] - [A_{yz}][A_{zz}]^{-1}[A_{zy}] \end{bmatrix}
$$

$$
[A_{xx}] = \sum_e \int\int_e [p_z\{U_y\}\{U_y\}^T - q_x k_0^2\{U\}\{U\}^T]\,dxdy
$$

presents a comparison of propagation constants for the first three modes in the slab dielectric loaded waveguide shown in Fig. 3. It can been seen from the table that the accuracy of the high-order mixed-edge rectangular elements is about one order higher than that of the lowest order one and have the same accuracy as that of the high-order covariant projection element—even if the unknowns are only 64% as many as those analyzed in the lowest order and the high-order covariant projection element. Fig. 4 presents the dispersion curves of the dominant mode in the waveguide partially filled with a lossy anisotropic dielectric block, with the real part of ϵ_{ry} as a parameter. The results calculated in [19] are also given in Fig. 4, indicated by dots. It can been seen from the curve that the agreement of the results is very good, and thus, the effectiveness and the accuracy of the present approach are verified.

IV. Conclusion

An efficient high-order mixed-edge rectangular element is proposed. It is demonstrated that the new element can provide higher calculating accuracy and efficiency with no spurious solutions. The matching relation among the orders of the interpolated polynomials for three components in the full vector functional in this paper is different from that in [18]. The success of the present element reveals that the matching relation given in this paper is an improvement to that given in [18]. This relationship plays an important role in eliminating spurious solutions and is the key point in increasing the accuracy and efficiency of the calculation.

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